Bilkent University
Department of Economics

Introduction to Vector Autoregression for Monetary Economics Research

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February 18, 2026

What are we doing?

- Main objective:
  - Measure effects of monetary policy on the economy: magnitude, timing, and persistence.

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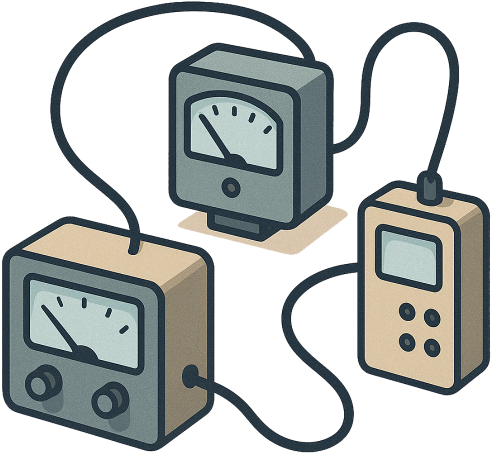
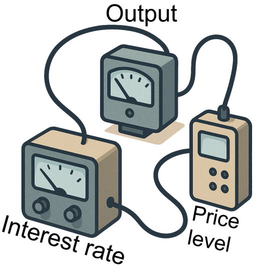
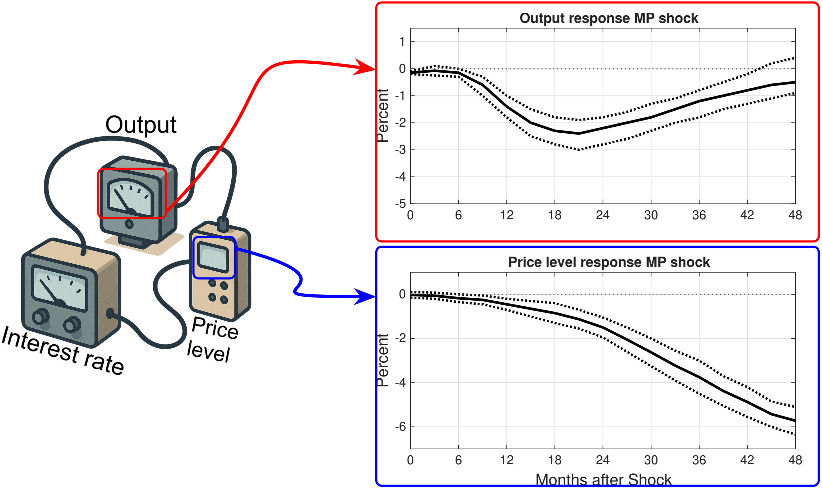
- Main objective:
  - Measure effects of monetary policy on the economy: magnitude, timing, and persistence.
- Key challenge:
  - Endogeneity problem - simultaneous causality between variables.
- A solution: **Vector Autoregression** approach.

What are we doing?



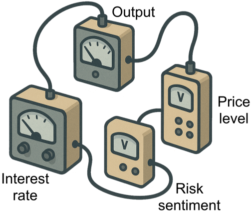
A model



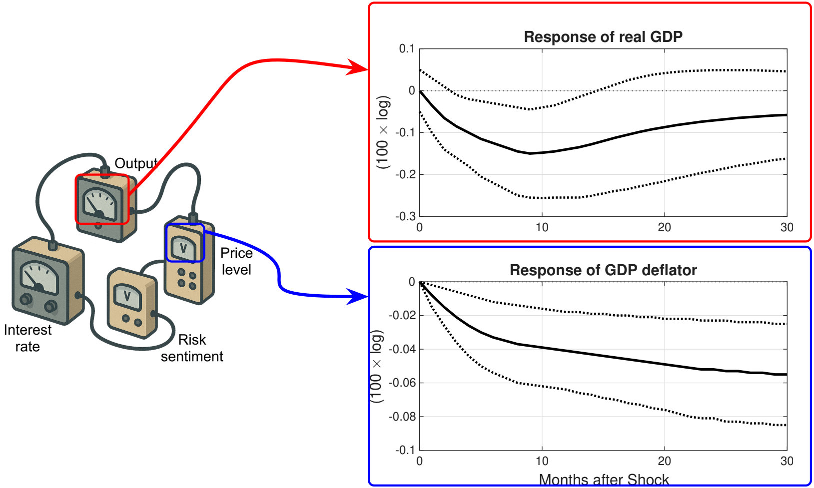


Output
Interest rate
Price level
Output response MP shock
Percent
1
0
-1
-2
-3
-4
-5
0 6 12 18 24 30 36 42 48
Price level response MP shock
Percent
0
-2
-4
-6
0 6 12 18 24 30 36 42 48
Months after Shock

Another model



Another model



What is autoregression anyway?

regress (n.)

late 14c., *regresse*, "a return, passage back, act of going back," from Latin *regressus* "a return, retreat, a going back," noun use of past participle of *regredi* "to go back," from re- "back" (see re-) + *gradi* "to step, walk" (from PIE root *ghredh- "to walk, go"). More common in legal language. Mental sense of "act of working back from an effect to a cause" is from 1610s.

also from late 14c.

Source: etymonline.com

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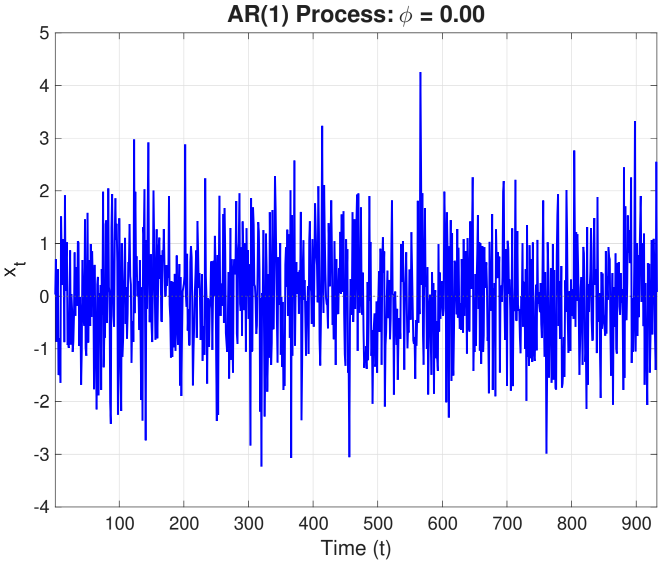
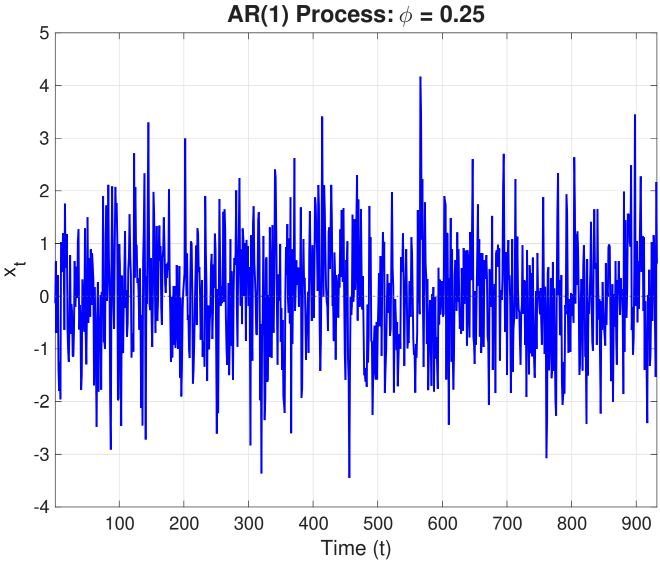
Usual regression: $y_t = \beta X_t + e_t$

Autoregression: $X_t = \phi X_{t-1} + w_t$

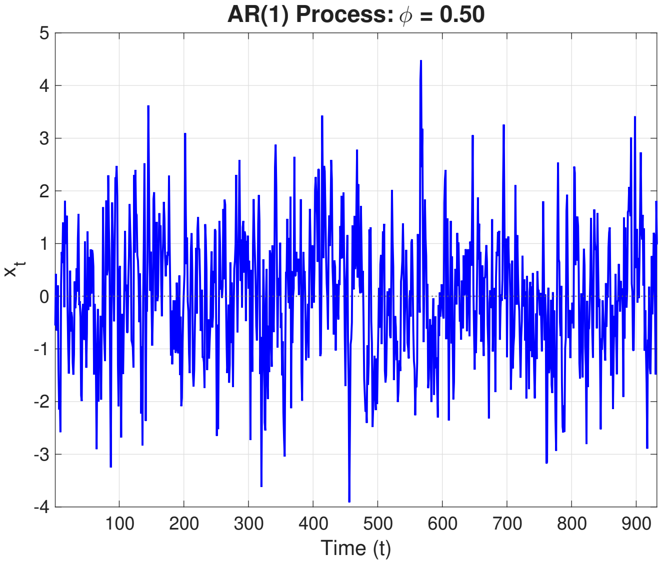
AR(1) process


AR(1) Process: ϕ = 0.00
5
4
3
2
1
0
-1
-2
-3
-4
x_t
100
200
300
400
500
600
700
800
900
Time (t)

AR(1) process

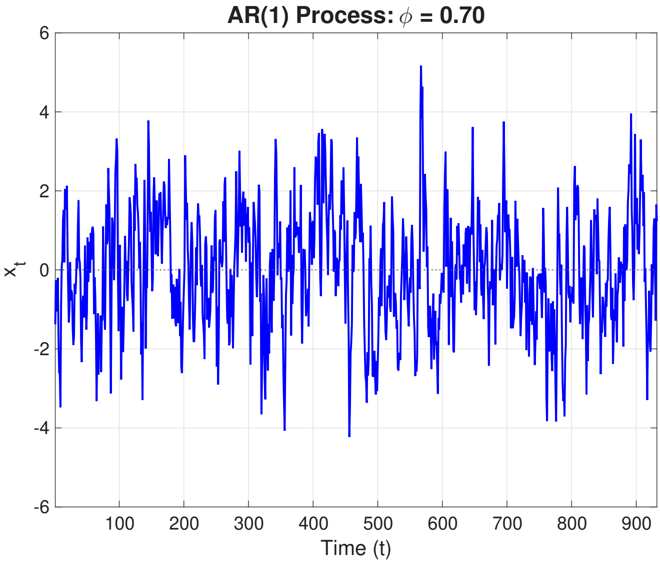

AR(1) Process: ϕ = 0.25
5
4
3
2
1
0
-1
-2
-3
-4
x_t
100
200
300
400
500
600
700
800
900
Time (t)

AR(1) process



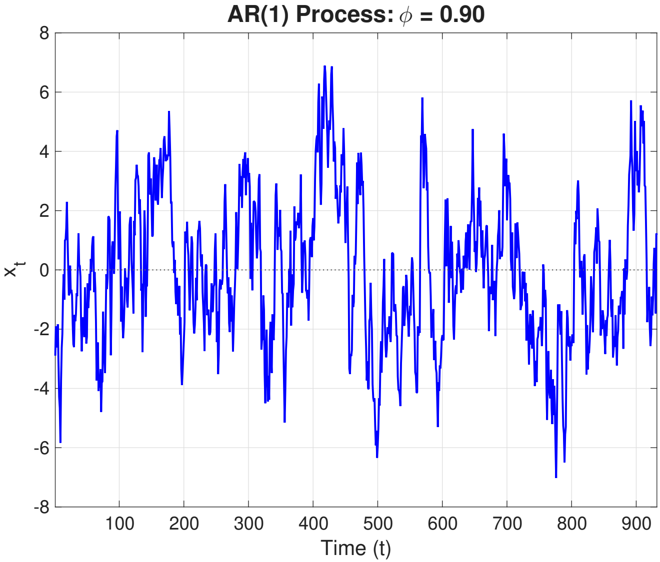
AR(1) Process: ϕ = 0.50
5
4
3
2
1
0
-1
-2
-3
-4
x_t
100
200
300
400
500
600
700
800
900
Time (t)

AR(1) process



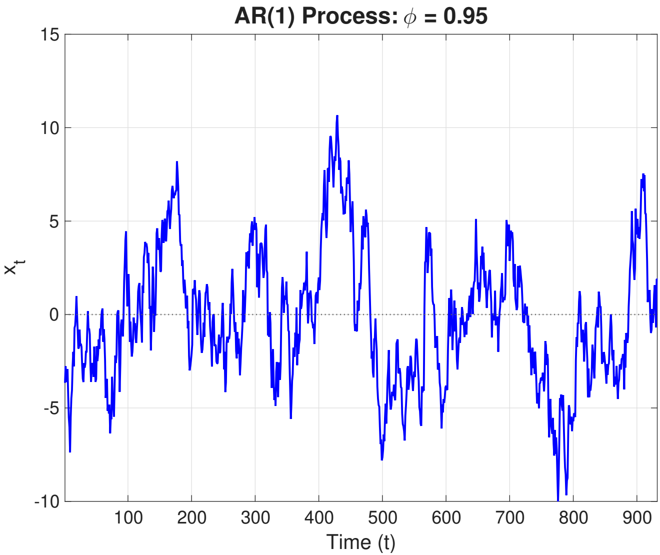
AR(1) Process: ϕ = 0.70
6
4
2
0
-2
-4
-6
x_t
100
200
300
400
500
600
700
800
900
Time (t)

AR(1) process

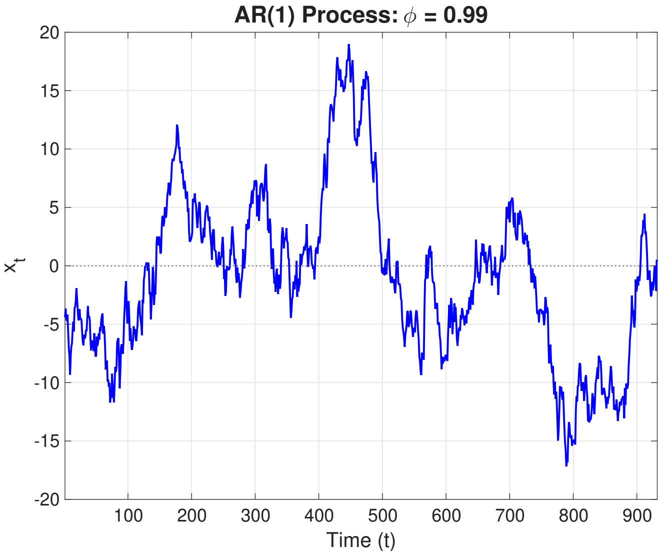


AR(1) Process: ϕ = 0.90
8
6
4
2
0
-2
-4
-6
-8
x_t
100
200
300
400
500
600
700
800
900
Time (t)

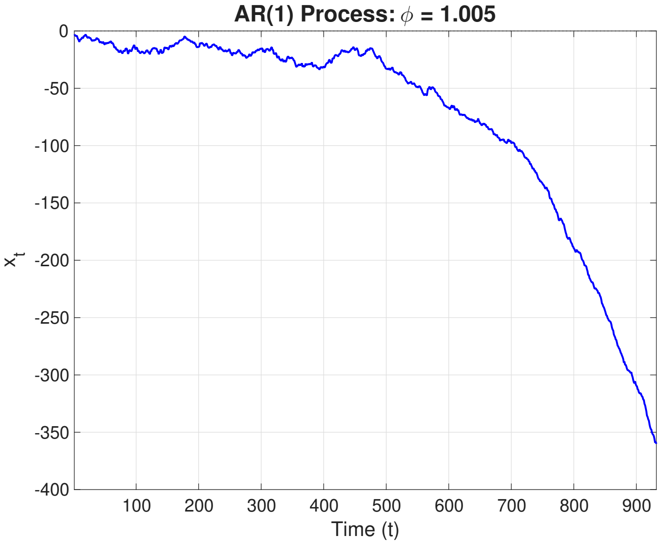
AR(1) process


AR(1) Process: ϕ = 0.95
15
10
5
0
-5
-10
x_t
100
200
300
400
500
600
700
800
900
Time (t)

AR(1) process

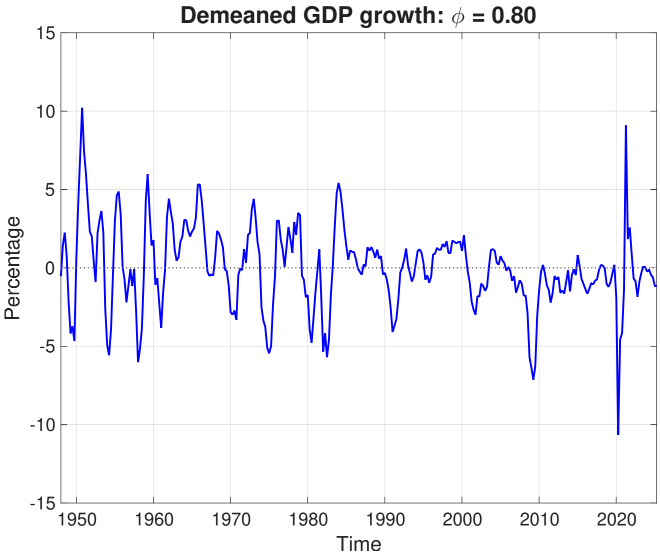

AR(1) Process: ϕ = 0.99
20
15
10
5
0
-5
-10
-15
-20
x_t
100
200
300
400
500
600
700
800
900
Time (t)

AR(1) process

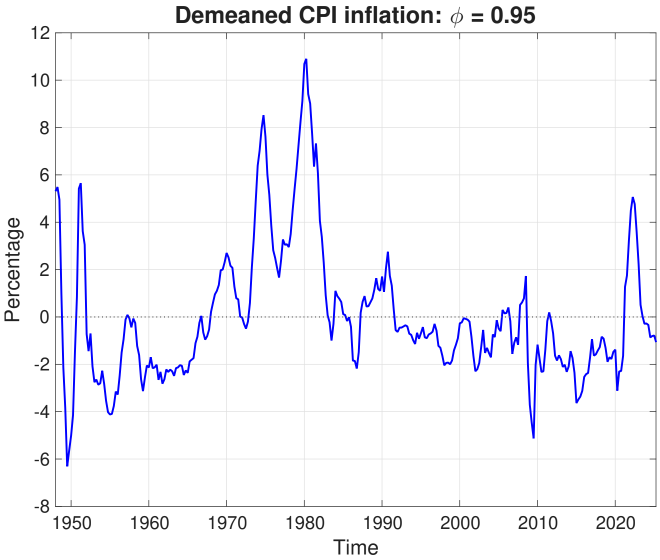


AR(1) Process: ϕ = 1.005
0
-50
-100
-150
-200
-250
-300
-350
-400
x_t
100
200
300
400
500
600
700
800
900
Time (t)

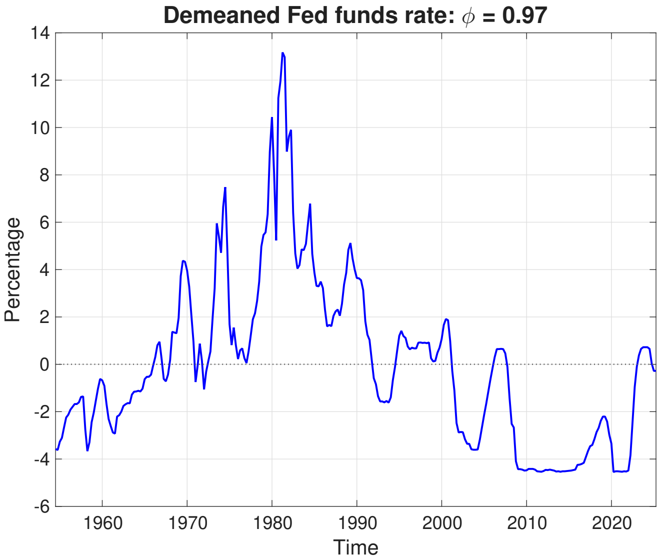
Real data example



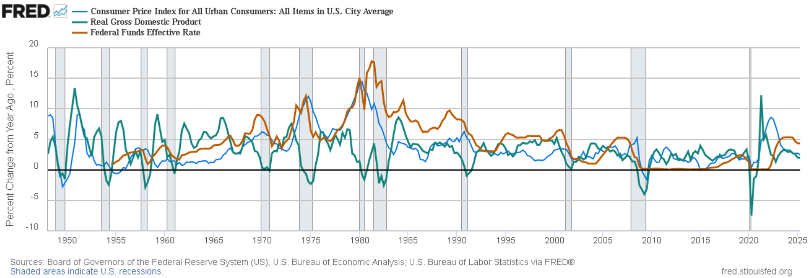
Real data example



Real data example



Interdependent economic variables



So, we can model these variables as autoregressive processes.

Interdependent economic variables

$$
\begin{aligned}
Y_t &= \phi_Y Y_{t-1} + u_{1,t} \quad \text{(ignores } \pi_t, i_t\text{)} \\
\pi_t &= \phi_\pi \pi_{t-1} + u_{2,t} \quad \text{(ignores } Y_t, i_t\text{)} \\
i_t &= \phi_i i_{t-1} + u_{3,t} \quad \text{(ignores } Y_t, \pi_t\text{)}
\end{aligned}
$$

Y_t should be $\Delta \log Y_t$ but I dropped $\Delta \log$ for brevity.

VAR ignoring ...

$$
\begin{aligned}
Y_t &= \phi_{11} Y_{t-1} + \phi_{12} \pi_{t-1} + \phi_{13} i_{t-1} + u_{1,t} \\
\pi_t &= \phi_{21} Y_{t-1} + \phi_{22} \pi_{t-1} + \phi_{23} i_{t-1} + u_{2,t} \\
i_t &= \phi_{31} Y_{t-1} + \phi_{32} \pi_{t-1} + \phi_{33} i_{t-1} + u_{3,t}
\end{aligned}
$$

VAR ignoring ...

$$x_t = \begin{bmatrix} Y_t \\ \pi_t \\ i_t \end{bmatrix} = \begin{bmatrix} \phi_{11} & \phi_{12} & \phi_{13} \\ \phi_{21} & \phi_{22} & \phi_{23} \\ \phi_{31} & \phi_{32} & \phi_{33} \end{bmatrix} \begin{bmatrix} Y_{t-1} \\ \pi_{t-1} \\ i_{t-1} \end{bmatrix} + \begin{bmatrix} u_{1,t} \\ u_{2,t} \\ u_{3,t} \end{bmatrix}$$

$$x_t = Ax_{t-1} + u_t$$

VAR ignoring ...

$$x_t = \begin{bmatrix} Y_t \\ \pi_t \\ i_t \end{bmatrix} = \begin{bmatrix} \phi_{11} & \phi_{12} & \phi_{13} \\ \phi_{21} & \phi_{22} & \phi_{23} \\ \phi_{31} & \phi_{32} & \phi_{33} \end{bmatrix} \begin{bmatrix} Y_{t-1} \\ \pi_{t-1} \\ i_{t-1} \end{bmatrix} + \begin{bmatrix} u_{1,t} \\ u_{2,t} \\ u_{3,t} \end{bmatrix}$$

$$x_t = Ax_{t-1} + u_t$$

Y_0, π_0, i_0 are the long run steady state levels, normalized to zero.

VAR ignoring ...

$$x_1 = Ax_0 + u_1 \qquad \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}_{x_1} = \underbrace{\begin{bmatrix} \phi_{11} & \phi_{12} & \phi_{13} \\ \phi_{21} & \phi_{22} & \phi_{23} \\ \phi_{31} & \phi_{32} & \phi_{33} \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}}_{x_0} + \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}_{u_1}$$

VAR ignoring ...

$$x_1 = Ax_0 + u_1 \qquad \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}_{x_1} = \underbrace{\begin{bmatrix} \phi_{11} & \phi_{12} & \phi_{13} \\ \phi_{21} & \phi_{22} & \phi_{23} \\ \phi_{31} & \phi_{32} & \phi_{33} \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}}_{x_0} + \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}_{u_1}$$

$$x_2 = Ax_1 \qquad \underbrace{\begin{bmatrix} \phi_{13} \\ \phi_{23} \\ \phi_{33} \end{bmatrix}}_{x_2} = \underbrace{\begin{bmatrix} \phi_{11} & \phi_{12} & \phi_{13} \\ \phi_{21} & \phi_{22} & \phi_{23} \\ \phi_{31} & \phi_{32} & \phi_{33} \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}_{x_1}$$

VAR ignoring ...

$$x_1 = Ax_0 + u_1 \qquad \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}_{x_1} = \underbrace{\begin{bmatrix} \phi_{11} & \phi_{12} & \phi_{13} \\ \phi_{21} & \phi_{22} & \phi_{23} \\ \phi_{31} & \phi_{32} & \phi_{33} \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}}_{x_0} + \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}_{u_1}$$

$$x_2 = Ax_1 \qquad \underbrace{\begin{bmatrix} \phi_{13} \\ \phi_{23} \\ \phi_{33} \end{bmatrix}}_{x_2} = \underbrace{\begin{bmatrix} \phi_{11} & \phi_{12} & \phi_{13} \\ \phi_{21} & \phi_{22} & \phi_{23} \\ \phi_{31} & \phi_{32} & \phi_{33} \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}_{x_1}$$

$$\cdots$$

$$x_t = A^{t-1}x_1 \qquad \underbrace{\begin{bmatrix} Y_t \\ \pi_t \\ i_t \end{bmatrix}}_{x_t} = A \underbrace{\begin{bmatrix} Y_t \\ \pi_t \\ i_t \end{bmatrix}}_{x_{t-1}}$$


NOPE
abc

VAR ignoring contemporaneous interactions

$$
\begin{aligned}
Y_t &= \phi_{11} Y_{t-1} + \phi_{12}\pi_{t-1} + \phi_{13} i_{t-1} + u_{1,t} \\
\pi_t &= \phi_{21} Y_{t-1} + \phi_{22}\pi_{t-1} + \phi_{23} i_{t-1} + u_{2,t} \\
i_t &= \phi_{31} Y_{t-1} + \phi_{32}\pi_{t-1} + \phi_{33} i_{t-1} + u_{3,t}
\end{aligned}
$$

VAR with contemporaneous interactions

Note: Minuses are only for easier exposition.

$$
\begin{aligned}
Y_t &= -b_{12}\pi_t - b_{13}i_t + \phi_{11}Y_{t-1} + \phi_{12}\pi_{t-1} + \phi_{13}i_{t-1} + \varepsilon_{Y,t} \\
\pi_t &= -b_{21}Y_t - b_{23}i_t + \phi_{21}Y_{t-1} + \phi_{22}\pi_{t-1} + \phi_{23}i_{t-1} + \varepsilon_{\pi,t} \\
i_t &= -b_{31}Y_t - b_{32}\pi_t + \phi_{31}Y_{t-1} + \phi_{32}\pi_{t-1} + \phi_{33}i_{t-1} + \varepsilon_{i,t}
\end{aligned}
$$

VAR with contemporaneous interactions

Note: Minuses are only for easier exposition.

$$Y_t = -b_{12}\pi_t - b_{13}i_t + \phi_{11}Y_{t-1} + \phi_{12}\pi_{t-1} + \phi_{13}i_{t-1} + \varepsilon_{Y,t}$$
$$\pi_t = -b_{21}Y_t - b_{23}i_t + \phi_{21}Y_{t-1} + \phi_{22}\pi_{t-1} + \phi_{23}i_{t-1} + \varepsilon_{\pi,t}$$
$$i_t = -b_{31}Y_t - b_{32}\pi_t + \phi_{31}Y_{t-1} + \phi_{32}\pi_{t-1} + \phi_{33}i_{t-1} + \varepsilon_{i,t}$$

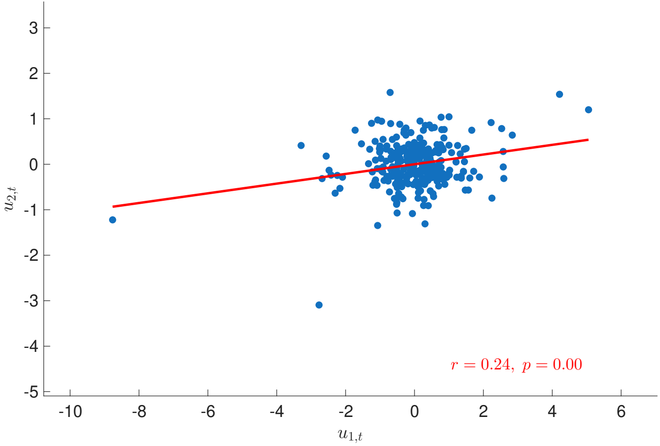
$$Y_t = \tilde{\phi}_{11}Y_{t-1} + \tilde{\phi}_{12}\pi_{t-1} + \tilde{\phi}_{13}i_{t-1} + \underbrace{-b_{12}\varepsilon_{\pi,t} - b_{13}\varepsilon_{i,t} + \varepsilon_{Y,t}}_{u_{1,t}}$$

$$i_t = \tilde{\phi}_{31}Y_{t-1} + \tilde{\phi}_{32}\pi_{t-1} + \tilde{\phi}_{33}i_{t-1} + \underbrace{-b_{31}\varepsilon_{Y,t} - b_{32}\varepsilon_{\pi,t} + \varepsilon_{i,t}}_{u_{2,t}}$$

$$\pi_t = \tilde{\phi}_{12}Y_{t-1} + \tilde{\phi}_{22}\pi_{t-1} + \tilde{\phi}_{23}i_{t-1} + \underbrace{-b_{21}\varepsilon_{Y,t} - b_{23}\varepsilon_{i,t} + \varepsilon_{\pi,t}}_{u_{3,t}}$$

These residuals are correlated with each other.

Correlation between GDP and inflation residuals



VAR with contemporaneous interactions

$$
\begin{aligned}
Y_t &= -b_{12}\pi_t - b_{13}i_t + \phi_{11}Y_{t-1} + \phi_{12}\pi_{t-1} + \phi_{13}i_{t-1} + \varepsilon_{Y,t} \\
\pi_t &= -b_{21}Y_t - b_{23}i_t + \phi_{21}Y_{t-1} + \phi_{22}\pi_{t-1} + \phi_{23}i_{t-1} + \varepsilon_{\pi,t} \\
i_t &= -b_{31}Y_t - b_{32}\pi_t + \phi_{31}Y_{t-1} + \phi_{32}\pi_{t-1} + \phi_{33}i_{t-1} + \varepsilon_{i,t}
\end{aligned}
$$

VAR with contemporaneous interactions

$$
\begin{aligned}
Y_t &= -b_{12}\pi_t - b_{13}i_t + \phi_{11}Y_{t-1} + \phi_{12}\pi_{t-1} + \phi_{13}i_{t-1} + \varepsilon_{Y,t} \\
\pi_t &= -b_{21}Y_t - b_{23}i_t + \phi_{21}Y_{t-1} + \phi_{22}\pi_{t-1} + \phi_{23}i_{t-1} + \varepsilon_{\pi,t} \\
i_t &= -b_{31}Y_t - b_{32}\pi_t + \phi_{31}Y_{t-1} + \phi_{32}\pi_{t-1} + \phi_{33}i_{t-1} + \varepsilon_{i,t}
\end{aligned}
$$

$$
\begin{aligned}
Y_t + b_{12}\pi_t + b_{13}i_t &= \phi_{11}Y_{t-1} + \phi_{12}\pi_{t-1} + \phi_{13}i_{t-1} + \varepsilon_{Y,t} \\
b_{21}Y_t + \pi_t + b_{23}i_t &= \phi_{21}Y_{t-1} + \phi_{22}\pi_{t-1} + \phi_{23}i_{t-1} + \varepsilon_{\pi,t} \\
b_{31}Y_t + b_{32}\pi_t + i_t &= \phi_{31}Y_{t-1} + \phi_{32}\pi_{t-1} + \phi_{33}i_{t-1} + \varepsilon_{i,t}
\end{aligned}
$$

Structural Vector Autoregression (SVAR)

$$\underbrace{\begin{bmatrix} 1 & b_{12} & b_{13} \\ b_{21} & 1 & b_{23} \\ b_{31} & b_{32} & 1 \end{bmatrix}}_{B_0} \underbrace{\begin{bmatrix} Y_t \\ \pi_t \\ i_t \end{bmatrix}}_{x_t} = B_1 \underbrace{\begin{bmatrix} Y_{t-1} \\ \pi_{t-1} \\ i_{t-1} \end{bmatrix}}_{x_{t-1}} + \underbrace{\begin{bmatrix} \varepsilon_{Y,t} \\ \varepsilon_{\pi,t} \\ \varepsilon_{i,t} \end{bmatrix}}_{\varepsilon_t}$$

Structural Vector Autoregression (SVAR)

$$\underbrace{\begin{bmatrix} 1 & b_{12} & b_{13} \\ b_{21} & 1 & b_{23} \\ b_{31} & b_{32} & 1 \end{bmatrix}}_{B_0} \underbrace{\begin{bmatrix} Y_t \\ \pi_t \\ i_t \end{bmatrix}}_{x_t} = B_1 \underbrace{\begin{bmatrix} Y_{t-1} \\ \pi_{t-1} \\ i_{t-1} \end{bmatrix}}_{x_{t-1}} + \underbrace{\begin{bmatrix} \varepsilon_{Y,t} \\ \varepsilon_{\pi,t} \\ \varepsilon_{i,t} \end{bmatrix}}_{\varepsilon_t}$$

$$B_0 x_t = B_1 x_{t-1} + \varepsilon_t$$

Key insight: B_0 captures contemporaneous relationships between variables

From SVAR to reduced form VAR

$$B_0 y_t = B_1 y_{t-1} + \varepsilon_t$$

$$y_t = \underbrace{A_1}_{B_0^{-1} B_1} y_{t-1} + B_0^{-1} \varepsilon_t$$

From SVAR to reduced form VAR

$$B_0 y_t = B_1 y_{t-1} + \varepsilon_t$$

$$y_t = \underbrace{A_1}_{B_0^{-1} B_1} y_{t-1} + B_0^{-1} \varepsilon_t$$

$$\underbrace{\begin{bmatrix} Y_t \\ \pi_t \\ i_t \end{bmatrix}}_{y_t} = A_1 \underbrace{\begin{bmatrix} Y_{t-1} \\ \pi_{t-1} \\ i_{t-1} \end{bmatrix}}_{y_{t-1}} + B_0^{-1} \underbrace{\begin{bmatrix} \varepsilon_{Y,t} \\ \varepsilon_{\pi,t} \\ \varepsilon_{i,t} \end{bmatrix}}_{\varepsilon_t}$$

Key transformation: Reduced form can be estimated, but the impact matrix requires identification to obtain the propagation mechanism.

Impulse response function: Methodological setup

$$\underbrace{\begin{bmatrix} Y_t \\ \pi_t \\ i_t \end{bmatrix}}_{x_t} = A_1 \underbrace{\begin{bmatrix} Y_{t-1} \\ \pi_{t-1} \\ i_{t-1} \end{bmatrix}}_{x_{t-1}} + \underbrace{\begin{bmatrix} \tilde{b}_{11} & \tilde{b}_{12} & \tilde{b}_{13} \\ \tilde{b}_{21} & \tilde{b}_{22} & \tilde{b}_{23} \\ \tilde{b}_{31} & \tilde{b}_{32} & \tilde{b}_{33} \end{bmatrix}}_{B_0^{-1}} \underbrace{\begin{bmatrix} \varepsilon_{Y,t} \\ \varepsilon_{\pi,t} \\ \boldsymbol{\varepsilon_{i,t}} \end{bmatrix}}_{\varepsilon_t}$$

Goal: Trace the effects of a one-unit shock to the monetary policy equation (interest rate shock)

Impulse response analysis: Period 0 (Baseline equilibrium)

$$\underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}}_{y_0} = A_1 \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}}_{y_{-1}} + B_0^{-1} \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}}_{\varepsilon_0}$$

Initial state: All variables at long-run equilibrium (normalized to zero)

Impulse response analysis: Period 1 (Initial shock impact)

$$\underbrace{\begin{bmatrix} Y_1 \\ \pi_1 \\ i_1 \end{bmatrix}}_{y_1} = A_1 \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}}_{y_0} + \underbrace{\begin{bmatrix} \tilde{b}_{11} & \tilde{b}_{12} & \boldsymbol{\tilde{b}_{13}} \\ \tilde{b}_{21} & \tilde{b}_{22} & \boldsymbol{\tilde{b}_{23}} \\ \tilde{b}_{31} & \tilde{b}_{32} & \boldsymbol{\tilde{b}_{33}} \end{bmatrix}}_{B_0^{-1}} \underbrace{\begin{bmatrix} 0 \\ 0 \\ \mathbf{1} \end{bmatrix}}_{\varepsilon_1}$$

Impact period: One-unit structural shock to interest rate equation

Impulse response analysis: Period 2 (First-order propagation)

$$\underbrace{\begin{bmatrix} Y_2 \\ \pi_2 \\ i_2 \end{bmatrix}}_{y_2} = A_1 \underbrace{\begin{bmatrix} Y_1 \\ \pi_1 \\ i_1 \end{bmatrix}}_{y_1} + B_0^{-1} \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}}_{\varepsilon_2}$$

Impulse response analysis: Period 2 (First-order propagation)

$$\underbrace{\begin{bmatrix} Y_2 \\ \pi_2 \\ i_2 \end{bmatrix}}_{y_2} = A_1 \underbrace{\begin{bmatrix} Y_1 \\ \pi_1 \\ i_1 \end{bmatrix}}_{y_1} + B_0^{-1} \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}}_{\varepsilon_2}$$

$$y_2 = A_1 \left[B_0^{-1} \underbrace{\begin{bmatrix} 0 \\ 0 \\ \mathbf{1} \end{bmatrix}}_{\varepsilon_1} \right]$$

First propagation: Effects transmitted through lagged relationships

Impulse response analysis: Period 3 (Second-order propagation)

$$y_3 = {A_1}^2 \left[B_0^{-1} \underbrace{\begin{bmatrix} 0 \\ 0 \\ \mathbf{1} \end{bmatrix}}_{\varepsilon_1} \right]$$

Second propagation: Compounding effects through dynamic interactions

General impulse response function

$$IRF_t^{(MP)} = {A_1}^{t-1} B_0^{-1} \underbrace{\begin{bmatrix} 0 \\ 0 \\ \mathbf{1} \end{bmatrix}}_{\text{MP shock}}$$

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Key properties:

- IRF depends on autoregressive coefficients (A_1) and impact matrix (B_0^{-1})
- Contemporaneous effects captured by B_0^{-1}
- Can trace response of any variable to any structural shock

$p > 1$ lags?

$$\text{Impulse:} \quad R_0 = B_0^{-1} \underbrace{\begin{bmatrix} 0 \\ 0 \\ \mathbf{1} \end{bmatrix}}_{\text{MP shock}}$$

$$\text{Response at } (t = 1): \quad R_1 = A_1 R_0$$

$$\text{Response at } (t = 2): \quad R_2 = A_1 R_1 + A_2 R_0$$

$$\text{Response at } (t = 3): \quad R_3 = A_1 R_2 + A_2 R_1 + A_3 R_0$$

$$\vdots$$

$$\text{Response at } (t = h > 0): \quad R_h = \sum_{i=1}^{\min\{h,p\}} A_i R_{h-i}$$

The identification problem

Key challenge: We can estimate the reduced form VAR, but how do we recover the structural parameters in B_0^{-1}?

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Problem:

- We observe u_t but not ε_t
- Need to identify B_0^{-1} to compute impulse responses
- With 3 variables: 9 elements in B_0^{-1}, but only 6 unique elements in $\text{Cov}(u_t)$ (Elaborated in the next slide)

Identification with 3 variables

Let

$$B_0^{-1} \equiv \tilde{B} = \begin{bmatrix} \tilde{b}_{11} & \tilde{b}_{12} & \tilde{b}_{13} \\ \tilde{b}_{21} & \tilde{b}_{22} & \tilde{b}_{23} \\ \tilde{b}_{31} & \tilde{b}_{32} & \tilde{b}_{33} \end{bmatrix}, \quad \varepsilon_t \sim \mathcal{N}(0, I_3) \quad u_t = \tilde{B}\varepsilon_t$$

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Then

$$\Sigma_u \equiv \text{Cov}(u_t) = \tilde{B}\tilde{B}^\top = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22} & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} \end{bmatrix}$$

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with entries (six unique equations):

$$\sigma_{11} = \tilde{b}_{11}^2 + \tilde{b}_{12}^2 + \tilde{b}_{13}^2 \qquad \sigma_{12} = \tilde{b}_{11}\tilde{b}_{21} + \tilde{b}_{12}\tilde{b}_{22} + \tilde{b}_{13}\tilde{b}_{23}$$

$$\sigma_{22} = \tilde{b}_{21}^2 + \tilde{b}_{22}^2 + \tilde{b}_{23}^2 \qquad \sigma_{13} = \tilde{b}_{11}\tilde{b}_{31} + \tilde{b}_{12}\tilde{b}_{32} + \tilde{b}_{13}\tilde{b}_{33}$$

$$\sigma_{33} = \tilde{b}_{31}^2 + \tilde{b}_{32}^2 + \tilde{b}_{33}^2 \qquad \sigma_{23} = \tilde{b}_{21}\tilde{b}_{31} + \tilde{b}_{22}\tilde{b}_{32} + \tilde{b}_{23}\tilde{b}_{33}$$

Identification with 3 variables

- Unknowns: the 9 numbers $\tilde{b}_{11}, \ldots, \tilde{b}_{33}$.
- Observables: the 6 numbers $\sigma_{11}, \sigma_{22}, \sigma_{33}, \sigma_{12}, \sigma_{13}, \sigma_{23}$.

Hence you need $9 - 6 = 3$ additional restrictions (e.g. long-run restrictions, zero and sign restrictions) to pin down B_0^{-1} and thus the impulse responses.

Cholesky factorization

Theorem (Cholesky Factorization)

Let $A \in \mathbb{R}^{n \times n}$ be symmetric and positive definite with strictly positive diagonal entries. Then there exists a unique lower triangular matrix $L \in \mathbb{R}^{n \times n}$ such that

$$A = LL^{\mathsf{T}}.$$

Cholesky identification: Economic intuition

Key idea: Order variables by speed of response to shocks

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Economic reasoning for monetary VAR:

- Output (Y): Responds slowly due to adjustment costs
- Inflation (π): Intermediate speed, some price stickiness
- Interest rate (i): Responds immediately to all shocks

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Ordering implication: $Y_t \to \pi_t \to i_t$

- Interest rate responds to output and inflation shocks *within the same period*
- Inflation responds to output shocks *within the same period*
- Output responds to other shocks only with a lag

Cholesky identification: Mathematical structure

Restriction: Make B_0 lower triangular

$$B_0 = \begin{bmatrix} 1 & 0 & 0 \\ b_{21} & 1 & 0 \\ b_{31} & b_{32} & 1 \end{bmatrix}$$

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Restriction: Make B_0 lower triangular

$$B_0 = \begin{bmatrix} 1 & 0 & 0 \\ b_{21} & 1 & 0 \\ b_{31} & b_{32} & 1 \end{bmatrix}$$

Structural system becomes:

$$\begin{aligned} Y_t &= \phi_{11} Y_{t-1} + \phi_{12}\pi_{t-1} + \phi_{13} i_{t-1} + \varepsilon_{Y,t} \\ \pi_t &= b_{21} Y_t + \phi_{21} Y_{t-1} + \phi_{22}\pi_{t-1} + \phi_{23} i_{t-1} + \varepsilon_{\pi,t} \\ i_t &= b_{31} Y_t + b_{32}\pi_t + \phi_{31} Y_{t-1} + \phi_{32}\pi_{t-1} + \phi_{33} i_{t-1} + \varepsilon_{i,t} \end{aligned}$$

Interpretation: Output doesn't respond contemporaneously to other variables' shocks

Cholesky decomposition implementation

Step 1: Estimate reduced form VAR to get $\hat{\Sigma}_u = E[u_t u_t']$
Step 2: Perform Cholesky decomposition

$$\hat{\Sigma}_u = B_0^{-1} B_0^{-1\mathsf{T}}$$

where B_0^{-1} is lower triangular

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Step 3: Identify structural shocks

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Step 4: Compute impulse responses

$$\mathrm{IRF}_t^{(k)} = A_1^{t-1} B_0^{-1} e_k$$

where e_k is the k-th unit vector

Cholesky example: Numerical illustration

Suppose we estimate the reduced form and obtain:

$$\hat{\Sigma}_u = \begin{bmatrix} 1.0 & 0.3 & 0.2 \\ 0.3 & 0.8 & 0.4 \\ 0.2 & 0.4 & 1.2 \end{bmatrix}$$

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Cholesky decomposition gives:

$$P = \begin{bmatrix} 1.0 & 0 & 0 \\ 0.3 & 0.84 & 0 \\ 0.2 & 0.40 & 1 \end{bmatrix}$$

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Interpretation of first column: A one-unit output shock causes:

- Output to increase by 1.0 contemporaneously
- Inflation to increase by 0.3 contemporaneously
- Interest rate to increase by 0.2 contemporaneously

Sensitivity to variable ordering

Different orderings $\Rightarrow$ Different economic assumptions
Alternative 1: Interest rate first (Taylor rule responds immediately)

$$i_t \to Y_t \to \pi_t$$

Alternative 2: Inflation first (price level responds to all shocks)

$$\pi_t \to Y_t \to i_t$$

Cholesky identification: advantages & limitations

Advantages:

- Simple to implement (built into most software)
- Always provides exact identification
- Useful benchmark for comparison
- Good starting point for empirical analysis

Limitations:

- Relies on strong (often unrealistic) assumptions about timing
- Results can be sensitive to variable ordering
- May not reflect true economic structure

Best practice: Use Cholesky as a baseline, then explore alternative identification schemes.

Proxy VAR

Remember the identification problem.

$$B_0^{-1} \equiv \tilde{B} = \begin{bmatrix} \tilde{b}_{11} & \tilde{b}_{12} & \tilde{b}_{13} \\ \tilde{b}_{21} & \tilde{b}_{22} & \tilde{b}_{23} \\ \tilde{b}_{31} & \tilde{b}_{32} & \tilde{b}_{33} \end{bmatrix}, \quad \varepsilon_t \sim \mathcal{N}(0, I_3) \quad u_t = \tilde{B}\varepsilon_t$$

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$$\sigma_{33} = \tilde{b}_{31}^2 + \tilde{b}_{32}^2 + \tilde{b}_{33}^2 \qquad \sigma_{23} = \tilde{b}_{21}\tilde{b}_{31} + \tilde{b}_{22}\tilde{b}_{32} + \tilde{b}_{23}\tilde{b}_{33}$$

Proxy VAR

Karel Mertens and Morten O. Ravn (2013).
"The Dynamic Effects of Personal and Corporate Income Tax Changes in the United States." *American Economic Review*, 103(4): 1212–1247.

Mark Gertler and Peter Karadi (2015).
"Monetary Policy Surprises, Credit Costs, and Economic Activity."
American Economic Journal: Macroeconomics, 7(1): 44–76.

Proxy VAR

$$IRF_t^{(MP)} = \underbrace{A_1{}^{t-1}}_{Estimated} \begin{bmatrix} \tilde{b}_{11} & \tilde{b}_{12} & \tilde{b}_{13} \\ \tilde{b}_{21} & \tilde{b}_{22} & \tilde{b}_{23} \\ \tilde{b}_{31} & \tilde{b}_{32} & \tilde{b}_{33} \end{bmatrix} \underbrace{\begin{bmatrix} \mathbf{1} \\ 0 \\ 0 \end{bmatrix}}_{\text{MP shock}} \quad \text{for VAR(1)}$$

We are only interested in responses to monetary policy shocks. So, to obtain IRFs, it is enough to identify the first column, $\left[\tilde{b}_{11} \;\; \tilde{b}_{21} \;\; \tilde{b}_{31}\right]'$.

The ordering of variables is inconsequential here; arranging the MPS first is just a convention.

External instrument assumptions

- Relevance: $\mathbb{E}\left[\varepsilon_{i,t} z_t\right] = \text{Cov}(\varepsilon_{i,t}, z_t) \neq 0$.
- Exogeneity (exclusion): $\text{Cov}(\varepsilon_{Y,t}, z_t) = 0, \ \text{Cov}(\varepsilon_{\pi,t}, z_t) = 0$.
- Instrument is uncorrelated with lagged information used to form u_t.

Then:

$$\mathbb{E}[u_t z_t] = \mathbb{E}\left[\underbrace{\tilde{B}\begin{bmatrix}\varepsilon_{i,t}\\ \varepsilon_{Y,t}\\ \varepsilon_{\pi,t}\end{bmatrix}}_{u_t} z_t\right] = \tilde{B}\begin{bmatrix}\text{Cov}(\varepsilon_{i,t}, z_t)\\ 0\\ 0\end{bmatrix} = \text{Cov}(\varepsilon_{i,t}, z_t) \ \begin{bmatrix}\tilde{b}_{i1}\\ \tilde{b}_{Y1}\\ \tilde{b}_{\pi1}\end{bmatrix}$$

$$\mathbb{E}\left[\begin{bmatrix} u_{i,t} \\ u_{Y,t} \\ u_{\pi,t} \end{bmatrix} z_t\right] = \text{Cov}(\varepsilon_{i,t}, z_t) \begin{bmatrix} \tilde{b}_{i1} \\ \tilde{b}_{Y1} \\ \tilde{b}_{\pi 1} \end{bmatrix}$$

Writing components,

$$\text{Cov}(u_{i,t}, z_t) = \text{Cov}(\varepsilon_{i,t}, z_t)\, \tilde{b}_{i1},$$

$$\text{Cov}(u_{Y,t}, z_t) = \text{Cov}(\varepsilon_{i,t}, z_t)\, \tilde{b}_{Y1},$$

$$\text{Cov}(u_{\pi,t}, z_t) = \text{Cov}(\varepsilon_{i,t}, z_t)\, \tilde{b}_{\pi 1}.$$

We normalize $\tilde{b}_{i1} = 1$, implying a one standard deviation shock to gauge the responses. Then, the first column is

$$\boxed{\tilde{b}_1 = \frac{\text{Cov}(u_t, z_t)}{\text{Cov}(u_{i,t}, z_t)}}$$

Wild bootstrap: Testing robustness of IRFs

Core question: Is my IRF a robust economic relationship or just a historical accident?

The method:

- Keep the same shock magnitudes and same timing
- Randomly flip the signs of historical residuals
- Create 1000+ alternative histories where shocks could have gone either way

The test:

- **Narrow confidence bands** ⇒ Robust economic relationship
- **Wide confidence bands** ⇒ Result depends on historical accidents

Intuition: If your IRF survives when we randomly flip whether shocks were positive or negative, then it captures something fundamental about the economy, not just the particular sequence of ups and downs that happened in your sample.

Wild bootstrapping method

Original VAR model:

$$Y_t = A_1 Y_{t-1} + A_2 Y_{t-2} + \cdots + A_p Y_{t-p} + u_t$$

Step 1: Extract residuals

$$u_t = Y_t - \hat{A}_1 Y_{t-1} - \hat{A}_2 Y_{t-2} - \cdots - \hat{A}_p Y_{t-p}$$

Step 2: Generate random signs

$$v_t = \begin{cases} +1 & \text{with probability } 0.5 \\ -1 & \text{with probability } 0.5 \end{cases}$$

Step 3: Create bootstrap residuals

$$u_t^* = v_t \cdot u_t$$

Key: Same shock magnitudes $|u_t| = |u_t^*|$, random signs v_t

Wild bootstrapping method

Step 4: Generate bootstrap sample

$$Y_t^* = A_1 Y_{t-1} + A_2 Y_{t-2} + \cdots + A_p Y_{t-p} + u_t^*$$

Step 5: Re-estimate

$$Y_t^* = \hat{A}_1 Y_{t-1}^* + \hat{A}_2 Y_{t-2}^* + \cdots + \hat{A}_p Y_{t-p}^* + u_t^*$$

Compute the impulse responses for this bootstrap sample. Repeat the procedure thousands of times and construct the 95% confidence interval from the resulting distribution.

Further reading

- *Structural Vector Autoregressive Analysis*
  by Lutz Kilian and Helmut Lütkepohl
- Identifying Structural VARs with a Proxy Variable and a Test for a Weak Proxy by Kurt G. Lunsford