Gürkaynak, Sack, Swanson (2005, IJCB) factor rotation

$$[mp1 \quad mp2 \quad ed2 \quad ed3 \quad ed4] = [F_1 \quad F_2 \quad F_3 \quad F_4 \quad F_5] \begin{bmatrix} \gamma_1 & \gamma_2 \\ \lambda_1 & \lambda_2 \\ \ldots & \end{bmatrix} + \eta \tag{1}$$

PCA gives $F = [F_1 \quad F_2]$, which is an uninterpretable $T \times 2$ matrix, and $F_1 \perp F_2$.

$$mp1 = \gamma_1 F_1 + \gamma_2 F_2 + \text{ignore } F_3, F_4, F_5 \tag{2}$$

We will rotate F to allow for a structural interpretation of its vectors.

$$\underbrace{\begin{bmatrix} Z_1 & Z_2 \end{bmatrix}}_{Z\text{: structural factors}} = F \underbrace{\begin{bmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \end{bmatrix}}_{U\text{: rotation matrix}} \tag{A8}$$

and where U is identified by certain restrictions.

First, the new factors Z_1 and Z_2 should remain orthogonal to each other:

$$\begin{aligned} E(Z_1 Z_2) &= (\alpha_1 F_1 + \alpha_2 F_2)(\beta_1 F_1 + \beta_2 F_2) = 0 \\ &= \alpha_1 \beta_1 \underbrace{\text{Var}(F_1^2)}_{:=\nu_{F_1}} + \underbrace{(\alpha_1 \beta_2 + \alpha_2 \beta_1) F_1 F_2}_{\text{zero as } F_1 \perp F_2} + \alpha_2 \beta_2 \underbrace{\text{Var}(F_2^2)}_{:=\nu_{F_2}} = 0 \\ &= \alpha_1 \beta_1 \nu_{F_1} + \alpha_2 \beta_2 \nu_{F_2} = 0 \end{aligned} \tag{A9}$$

Also, we impose the restriction that Z_2 does not influence the current policy surprise, $mp1$, as follows. That is to say, if we write $mp1$ in terms of Z_1 and Z_2, the coefficient of Z_2 should be zero.

We can already write $mp1$ in terms of F_1 and F_2,

$$mp1_t = \gamma_1 F_1 + \gamma_2 F_2 \quad + \text{ ignore } F_3, F_4, F_5 \tag{3}$$

where γ_1 and γ_2 are known loadings on $mp1$. As $F = ZU^{-1}$, we can write F_1 and F_2 in terms of Z_1 and Z_2

$$F_1 = \frac{1}{\alpha_1 \beta_2 - \alpha_2 \beta_1} \left[\beta_2 Z_1 - \alpha_2 Z_2 \right] \tag{A10}$$

$$F_2 = \frac{1}{\alpha_1 \beta_2 - \alpha_2 \beta_1} \left[\alpha_1 Z_2 - \beta_1 Z_1 \right] \tag{A11}$$

as $U^{-1} = \frac{1}{\alpha_1\beta_2 - \alpha_2\beta_1} \begin{bmatrix} \beta_2 & -\beta_1 \\ -\alpha_2 & \alpha_1 \end{bmatrix}$

Now, plug (A10) and (A11) into Equation (3).

$$mp1_t = \gamma_1 \frac{1}{\alpha_1\beta_2 - \alpha_2\beta_1}\left[\beta_2 Z_1 - \alpha_2 Z_2\right] + \gamma_2 \frac{1}{\alpha_1\beta_2 - \alpha_2\beta_1}\left[\alpha_1 Z_2 - \beta_1 Z_1\right]$$

The coefficient of Z_2 is equal to

$$\frac{1}{\alpha_1\beta_2 - \alpha_2\beta_1}\left(-\gamma_1\alpha 2 + \gamma_2\alpha_1\right)$$

and we want it to be zero. It follows that

$$\gamma_2\alpha_1 - \gamma_1\alpha_2 = 0$$

This restriction pins down the *direction* of (α_1, α_2) (up to scale), i.e., it determines the orientation of the first rotated factor Z_1 in the space spanned by (F_1, F_2).

Given this direction for Z_1, the orthogonality condition (A9) determines the direction of (β_1, β_2), thereby fixing the orthogonal complement Z_2 (again up to scale). Thus, the rotation from (F_1, F_2) to (Z_1, Z_2) is fully determined by the identifying restriction that Z_2 does not affect $mp1$ and by the requirement that the rotated factors remain orthogonal.

Interpreting Z_2 as a forward guidance factor is, therefore, not an additional assumption but rather an empirical characterization of the orthogonal residual factor obtained from this rotation.

$$\gamma_2\alpha_1 - \gamma_1\alpha_2 = 0 \implies \frac{\alpha_1}{\alpha_2} = \frac{\gamma_1}{\gamma_2} \qquad \text{(A10 \& A11: mp1 restriction)}$$

$$\alpha_1\beta_1\nu_{F_1} + \alpha_2\beta_2\nu_{F_2} = 0 \implies \frac{\beta_1}{\beta_2} = \frac{-\alpha_2\nu_{F_2}}{\alpha_1\nu_{F_1}} \qquad \text{(A9: orthogonality restriction)}$$

We only need additional constraints on the magnitudes of these vectors, which will affect only the variances of the transformed vectors, not the direction of the vectors. This is acceptable, as we will later apply an appropriate rescaling.

$$(\alpha_1, \alpha_2) = c_1(\gamma_1, \gamma_2)$$

$$(\beta_1, \beta_2) = c_2(-\alpha_2\nu_{F_2}, \alpha_1\nu_{F_1})$$

Pick $\alpha_1 + \alpha_2 = 1 \Longrightarrow c_1 = \frac{1}{\gamma_1+\gamma_2}$. Then,

$$(\beta_1, \beta_2) = c_2 \left(\frac{-\gamma_2 \nu_{F_2}}{\gamma_1 + \gamma_2}, \frac{\gamma_1 \nu_{F_1}}{\gamma_1 + \gamma_2} \right)$$

Now, pick $b_1 + b_2 = 1$. Then,

$$c_2 = \frac{\gamma_1 + \gamma_2}{\gamma_1 \nu_{F_1} - \gamma_2 \nu_{F_2}}$$

So, we identify a unique U matrix.

$$U = \begin{bmatrix} \frac{\gamma_1}{\gamma_1+\gamma_2} & \frac{-\gamma_2 \nu_{F_2}}{\gamma_1 \nu_{F_1} - \gamma_2 \nu_{F_2}} \\ \frac{\gamma_2}{\gamma_1+\gamma_2} & \frac{\gamma_1 \nu_{F_1}}{\gamma_1 \nu_{F_1} - \gamma_2 \nu_{F_2}} \end{bmatrix}$$

We then rescale Z_1 so that Z_1 moves the current policy surprise $mp1$ one-for-one, and Z_2 has the same magnitude effect on the year-ahead eurodollar futures rate as Z_1 has on that rate.